1) Let $X \sim \mathrm{Uniform}(\{1,2,3\})$. Find the MGF of centralised X.

For centralised X,

$E[X] = \left(1 \times \frac{1}{3}\right) + \left(2 \times \frac{1}{3}\right) + \left(3 \times \frac{1}{3}\right)$

$\Rightarrow \frac{6}{3} = 2$

$Y = X - E[X]$

$\Rightarrow Y \sim \mathrm{Uniform}\{\overset{1/3}{-1}, \overset{1/3}{0}, \overset{1/3}{1}\}$

$M_X(\lambda) = E[e^{\lambda x}] = e^{-\lambda} \times \frac{1}{3} + e^{0\lambda} \times \frac{1}{3} + e^{\lambda} \times \frac{1}{3}$

$\Rightarrow \frac{e^{-\lambda} + 1 + e^{\lambda}}{3}$

2) Let $X_1, X_2, X_3 \sim$ iid X and $X \sim \mathrm{Uniform}(\{-0.5, 0.5\})$. Define $S = X_1 + X_2 + X_3$. Find the MGF of S.

'S' can take only 0.5 or -0.5 as values.

When iid variables are added, the MGF of the sum is simply the product of MGF of iid variables.

Short Proof,

MGF of X $= E[e^{\lambda x}] = \frac{1}{2} \times e^{-0.5\lambda} + \frac{1}{2} \times e^{0.5\lambda}$

MGF of S $= E[e^{\lambda(S)}] = E[e^{\lambda(X_1 + X_2 + X_3)}] = E[e^{\lambda X_1} \cdot e^{\lambda X_2} \cdot e^{\lambda X_3}]$

$\Rightarrow \left(E[e^{\lambda X_1}]\right)^3$ [X_1, X_2, X_3 are iid]

$\Rightarrow \left(\frac{e^{-0.5\lambda} + e^{0.5\lambda}}{2}\right)^3$

3) Let $X_1, X_2 \sim$ iid X and X be a discrete random variable with following probability mass function

$$P(X=k)=\begin{cases}0.2 & \text{for } k=-6\\ 0.4 & \text{for } k=1\\ 0.4 & \text{for } k=2\\ 0 & \text{otherwise.}\end{cases}$$

Define $Y = X_1 + X_2$. Find the distribution of Y.

Possible values of Y are $-5, -4, 2, 3, 4, -12$

$P(Y=-5) = P(X_1=-6, X_2=1) + P(X_1=1, X_2=-6)$

$\Rightarrow 0.2 \times 0.4 + 0.4 \times 0.2$

$\Rightarrow 0.16$

$P(Y=-4) = P(X_1=-6, X_2=2) + P(X_1=2, X_2=-6)$

$\Rightarrow 0.2 \times 0.4 + 0.4 \times 0.2$

$\Rightarrow 0.16$

$P(Y=2) = P(X_1=1, X_2=1)$

$\Rightarrow 0.4 \times 0.4$

$\Rightarrow 0.16$

$P(Y=3) = P(X_1=2, X_1=1) + P(X_1=1, X_2=1)$

$\Rightarrow 0.4 \times 0.4 + 0.4 \times 0.4$

$\Rightarrow 0.32$

$P(Y=4) = P(X_1=2, X_2=2)$

$\Rightarrow 0.4 \times 0.4$

$\Rightarrow 0.16$

$P(Y=-12) = P(X_1=-6, X_2=-6)$

$\Rightarrow 0.2 \times 0.2$

$\Rightarrow 0.04$

6) What is the value of sixth moment of the Normal$(0, 3)$?

Sixth moment of Z i.e. Normal $(0,1)$ is given as

$E[Z^{2m}] = (2m-1)\,E[Z^{2(m-1)}]$

This is done recursively till we reach $E[Z^2]$ $\quad [E[Z^2] = 1]$

$\Rightarrow E[Z^6] = (2\times3-1)\, E[Z^{2(3-1)}]$ $\quad [2m = 6]$

$\Rightarrow 5 \times E[Z^4]$

$\Rightarrow 5 \times (2\times2-1)\, E[Z^{2(2-1)}]$ $\quad [2m = 4]$

$\Rightarrow 5 \times 3 \times 1$

$\Rightarrow 15$

Normal(0,3) can be as Normal of $Z\sigma + \mu$

Here, $\mu = 0$

$\sigma = \sqrt{3}$

$\Rightarrow Z \times \sqrt{3}$

6^{th} Moment of Normal $(0,3)$

$\Rightarrow E[(\sqrt{3}Z)^6] = 3^3 \times E[Z^6]$

$\Rightarrow 27 \times 15 = \underline{\underline{405}}$