1) The joint PDF of two random variables X and Y is given by

$$f_{XY}(x,y) = \begin{cases} 6x^2y & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Calculate $P(Y > 0.5 \mid X = 0.5)$.

$$P(Y > 0.5 \mid X = 0.5) = \frac{f_{XY}(X=0.5, Y>0.5)}{f_X(0.5)}$$

$$f_X(x) = \int_0^1 6x^2 y \cdot dy$$

$$\Rightarrow 6x^2 \left[\frac{y^2}{2}\right]_0^1$$

$$\Rightarrow 3x^2$$

$$f_X(3.5) = 3 \times (0.5) \times (0.5)$$

$$\Rightarrow 0.75$$

$$f_{XY} = \int_{0.5}^{1} 6(0.5)^2 y \cdot dy$$

$$\Rightarrow f_{XY} = 1.5 \left[\frac{y^2}{2}\right]$$

$$\Rightarrow f_{XY} = 1.5 \left[\frac{1}{2} - \frac{(0.5)^2}{2}\right]$$

$$\Rightarrow f_{XY} = \frac{1.5}{2}[1 - 0.25] = 0.5625$$

$$\therefore P(Y > 0.5 \mid X = 0.5) = \frac{f_{XY}(X = 0.5, Y > 0.5)}{f_X(X = 0.5)} = \frac{0.5625}{0.75} = 0.75$$

2) Let the joint PDF of two random variables X and Y be given by

$$f_{XY}(x,y) = \begin{cases} \frac{x(1+2y)}{4} & \text{for } 0 < x < 2, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Calculate $P\left(\frac{1}{8} < x < \frac{1}{4} \mid y = \frac{1}{4}\right)$. Enter the answer correct to two decimal places.

$$P(1/8 < x < 1/4 \mid y = 1/4) = \frac{f_{XY}(1/8 < x < 1/4, 1/4)}{f_Y(1/4)}$$

$$f_{XY}(1/8 < x < 1/4, 1/4) = \int_{1/8}^{1/4} \frac{x(1 + 2(\frac{1}{4}))}{4} \cdot dx \qquad [y = 1/4]$$

$$\Rightarrow \frac{1}{4}(1 + 1/2) \int_{1/8}^{1/4} x \cdot dx$$

$$\Rightarrow \frac{3}{8}\left[\frac{x^2}{2}\right]_{1/8}^{1/4}$$

$$\Rightarrow \frac{3}{\ } \left[\frac{1}{\ } - \frac{1}{\ }\right]$$

$$16 \quad \left[\; 16 \qquad 64 \; \right]$$

$$\Rightarrow \quad \frac{3}{16_4} \times \frac{\cancel{4}}{64} = \frac{3}{256}$$

$$f_Y(y) = \int_0^2 \frac{x(1+2y)}{4} \cdot dx$$

$$\Rightarrow \left(\frac{1+2y}{4}\right) \int_0^2 x \cdot dx$$

$$\Rightarrow \left(\frac{1+2y}{4}\right) \left[\frac{x^2}{2}\right]_0^2$$

$$\Rightarrow \left(\frac{1+2y}{4}\right) \left[\frac{4}{2} - 0\right]$$

$$\Rightarrow \frac{1+2y}{2}$$

$$f_Y(1/4) = \frac{1+2(1/4)}{2} = \frac{3/2}{2} = \frac{3}{4}$$

$$\therefore P(1/8 < x < 1/4 \mid y = 1/4) = \frac{f_{XY}(1/8 < x < 1/4 \, , \, 1/4)}{f_Y(1/4)}$$

$$\Rightarrow \frac{3/256}{3/4} = \frac{4}{256}$$

3) Let $(X, Y) \sim \text{Uniform}(D)$, where $D := \{3X > Y, 0 < x < 1, y > 0\}$. Choose the correct option(s) from the following:

$(Y \mid X = a) \sim \text{Uniform}[0, 3a]$

We know,

$Y < 3X$

$\Rightarrow Y < 3a \qquad [X = a]$

Also, $0 < y$

$\Rightarrow$ Range of Y will be $0 < Y < 3a$

$(X \mid Y = b) \sim \text{Uniform}(b/3, 1)$

We know,

$3X > Y$

$\Rightarrow \quad X > \frac{b}{3} \qquad [Y = b]$

Also, $0 < x < 1$

$\Rightarrow$ Range of X will be $b/3 < X < 1$

4) Let $(X, Y) \sim \text{Uniform}(D)$, where $D := [0, 2] \times [2, 4]$. Choose the correct option(s) from the following:

$X \sim [0, 2]$ and $Y \sim [2, 4]$

$(Y \mid X = a) \sim \text{Uniform}[2, 4], \quad 0 < a < 2 \qquad [x = a]$

It will belong to range of Y because value of 'x' is given to be 'a'.
Hence, we will have to integrate the conditional distribution over the range of Y to get the density.

Similarly,
$(X \mid Y = b) \sim \text{Uniform}[0, 2], \quad 2 < b < 4$

5) Let the joint PDF of two random variables X and Y be given by

$$f_{XY}(x, y) = \begin{cases} ye^{-y(x+1)} & \text{for } x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of $P(X \mid Y = 1)$.

$$P(X \mid Y = 1) = \frac{f_{XY}(X, 1)}{f_Y(1)}$$

$$f_{XY}(X, 1) = (1)\, e^{-(1)(x+1)} \qquad [y = 1]$$

$$\Rightarrow e^{-x-1}$$

$$f_Y(y) = \int_0^\infty y e^{-y(x+1)} \cdot dx$$

$$\Rightarrow y \left[\frac{e^{-yx-y}}{-y} \right]_0^\infty$$

$$\Rightarrow -1 \left[e^{-\infty} - e^{-y} \right]$$

$$\Rightarrow e^{-y}$$

$$f_Y(1) = e^{-1}$$

$$\therefore \quad P(X \mid Y=1) = \frac{f_{XY}(x, 1)}{f_Y(1)}$$

$$\Rightarrow \quad \frac{e^{-x-1}}{e^{-1}} = \underline{\underline{e^{-x}}}$$