$2e^{-2x}$

Let $X \sim \text{Exp}(2)$ and $Y \sim \text{Exp}(4)$ be independent.

1) Find $f_{XY}(x, y)$.

$X \sim Exp(2)$ $\qquad$ $Y \sim Exp(4)$

$\Rightarrow X \sim [2e^{-2x}]$ $\qquad$ $Y \sim [4e^{-2y}]$

$$f_{XY}(x,y) = 2e^{-2x} \times 4e^{-4y}$$
$$\Rightarrow 8e^{-2x + -4y}$$
$$\Rightarrow 8e^{-2(x+2y)}$$

2) Find $P(X \leq 2Y)$.

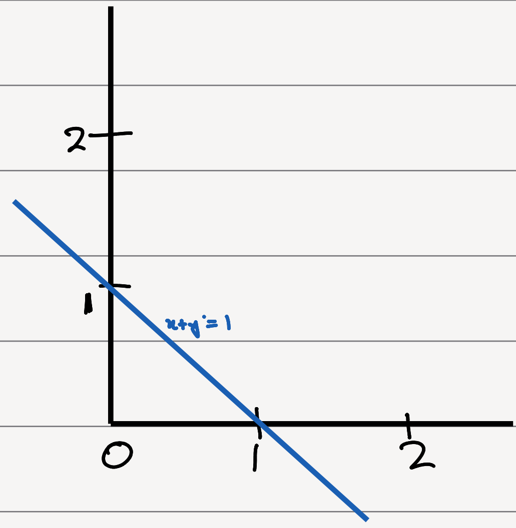
$$P(X \leq 2Y) = \int_0^4 \int_0^{2y} 8e^{-2(x+2y)} \cdot dx \cdot dy$$
$$\Rightarrow 8\int_0^4 \int_0^{2y} e^{-2x-4y}$$
$$\Rightarrow 8\int_0^4 \left[\frac{e^{-2x-4y}}{-2}\right]_0^{2y}$$
$$\Rightarrow -4\int_0^4 \left[e^{-2(2y)-4y} - e^{-2(0)-4y}\right]$$
$$\Rightarrow -4\int_0^4 e^{-8y} - e^{-4y}$$
$$\Rightarrow -4\left[\frac{e^{-8y}}{-8} - \frac{e^{-4y}}{-4}\right]_0^4$$
$$\Rightarrow -4\left[\frac{e^{-32}}{-8} - \frac{e^{-16}}{-4} - \left(\frac{e^0}{-8} - \frac{e^0}{-4}\right)\right]$$
$$\Rightarrow -4\left[\frac{e^{-16}}{4} - \frac{e^{-32}}{8} - \left(\frac{1}{4} - \frac{1}{8}\right)\right]$$
$$\Rightarrow -4\left[\frac{8e^{-16} - 4e^{-32}}{32} - \frac{4}{32}\right]$$

$[8e^{-16} - 4e^{-32} \simeq 0]$

$$\Rightarrow -4\left[\frac{-4}{32}\right]$$
$$\Rightarrow \underline{\underline{\frac{1}{2}}}$$

3) Let $(X, Y) \sim \text{Uniform}(D)$, where $D := \{(x, y) : x + y < 1, x > 0, y > 0\}$. Are X and Y independent?

- Yes
- No



$$f_{XY}(x,y) = \begin{cases} \frac{1}{2} & x>0, y>0, x+y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = \begin{cases} \int_0^1 1-y & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \begin{cases} \left[y - \frac{y^2}{2}\right]_0^1 = \frac{1}{2} & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_0^1 1-x & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{1}{2} & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{XY}(x,y) = \frac{1}{2}$$
$$f_X(x) * f_Y(y) = \frac{1}{2} \times \frac{1}{2}$$

$$\therefore f_{XY}(x,y) \neq f_X(x) * f_Y(y)$$

4) Let the joint PDF of two random variables X and Y be given by $f_{XY}(x, y) = \begin{cases} e^{-(x+y)} & \text{for } x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$

$$f_X(x) = \int_0^\infty e^{-(x+y)} \, dy$$

$$\Rightarrow \left[\frac{e^{-x-y}}{-1}\right]_0^\infty$$

$$\Rightarrow \left[\frac{e^{-\infty}}{-1} - \frac{e^{-x}}{-1}\right] = \underline{\underline{e^{-x}}}$$

$$\left[e^{-\infty} \simeq 0\right]$$

$$f_Y(y) = \int_0^{\infty} e^{-(x+y)} \cdot dx$$

$$\Rightarrow \left[\frac{e^{-x-y}}{-1}\right]_0^{\infty}$$

$$\Rightarrow \left[\frac{e^{-\infty}}{-1} - \frac{e^{-y}}{-1}\right] = \underline{\underline{e^{-y}}}$$

$$f_X(x) \times f_Y(y) = e^{-x} \times e^{-y}$$

$$\Rightarrow e^{-(x+y)}$$

$$\therefore f_{XY}(x,y) = f_X(x) \times f_Y(y)$$

$$P(X < 2) = \int_0^{\infty}\int_0^{2} e^{-(x+y)} \cdot dx \cdot dy$$

$$\Rightarrow \int_0^{\infty} \left[\frac{e^{-x-y}}{-1}\right]_0^2 \cdot dy$$

$$\Rightarrow \int_0^{\infty} \left[\frac{e^{-2-y}}{-1} - \frac{e^{-y}}{-1}\right] \cdot dy$$

$$\Rightarrow \int_0^{\infty} (e^{-y} - e^{-2-y}) \cdot dy$$

$$\Rightarrow \left[\frac{e^{-y}}{-1} - \frac{e^{-2-y}}{-1}\right]_0^{\infty}$$

$$\Rightarrow \left[e^{-2-y} - e^{-y}\right]_0^{\infty}$$

$$\Rightarrow \left[(e^{-\infty} - e^{-\infty}) - (e^{-2} - e^{0})\right]$$

$$\Rightarrow \left[0 - (e^{-2} - 1)\right]$$

$$\Rightarrow \underline{\underline{1 - e^{-2}}}$$

$$P(X < Y) = \int_0^{\infty}\int_0^{y} e^{-x-y} \cdot dx \cdot dy$$

$$\Rightarrow \int_0^{\infty} \left[\frac{e^{-x-y}}{-1}\right]_0^y \cdot dy$$

$$\Rightarrow \int_0^{\infty} \left[\frac{e^{-2y}}{-1} - \frac{e^{-y}}{-1}\right] \cdot dy$$

$$\Rightarrow \int_0^{\infty} (e^{-y} - e^{-2y}) \cdot dy$$

$$\Rightarrow \left[\frac{e^{-y}}{-1} - \frac{e^{-2y}}{-2} \right]_0^{\infty}$$

$$\Rightarrow \left[\frac{e^{-2y}}{2} - e^{-y} \right]_0^{\infty}$$

$$\Rightarrow \left[\left(\frac{e^{-\infty}}{2} - e^{-\infty} \right) - \left(\frac{e^0}{2} - e^0 \right) \right]$$

$$\Rightarrow \left[0 - \left(\tfrac{1}{2} - 1 \right) \right]$$

$$\Rightarrow \frac{1}{2}$$

5) Let the joint PDF of two random variables X and Y be given by

$$f_{XY}(x,y) = \begin{cases} \frac{4}{3}xy(1-x) & \text{for } 0 < x < 1, 0 < y < 3 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = \int_0^3 \frac{4}{3} xy(1-x) \cdot dy$$

$$\Rightarrow \frac{4}{3} x(1-x) \int_0^3 y \cdot dy$$

$$\Rightarrow \frac{4}{3} x(1-x) \left[\frac{y^2}{2} \right]_0^3$$

$$\Rightarrow \frac{4}{3} x(1-x) \left[\frac{9}{2} - 0 \right]$$

$$\Rightarrow \frac{\cancel{4}^{2}}{\cancel{3}_{1}} \times \frac{\cancel{9}^{3}}{\cancel{2}_{1}} (x)(1-x)$$

$$\Rightarrow 6(x)(1-x)$$

$$f_Y(y) = \int_0^1 \frac{4}{3} xy(1-x) \cdot dx$$

$$\Rightarrow \frac{4}{3} y \int_0^1 (x - x^2) \cdot dx$$

$$\Rightarrow \frac{4}{3} y \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$$

$$\Rightarrow \frac{4}{3} y \left[\frac{1}{2} - \frac{1}{3} - (0-0) \right]$$

$$\Rightarrow \frac{\cancel{4}^{2}}{3} \times \frac{1}{\cancel{6}_{3}} \times y$$

$$\Rightarrow \frac{2}{9} y$$

$$f_X(x) \times f_Y(y) = 6(x)(1-x) \times \frac{2}{9} y$$

$$\Rightarrow \frac{12}{9} yx(1-x)$$

$$\Rightarrow \frac{4}{3} yx(1-x)$$

$$\therefore f_X(x) \times f_Y(y) = f_{XY}(x,y)$$