Let X and Y have the joint density

$$f_{XY}(x,y) = \begin{cases} \frac{1}{3} & \text{for } 1 \le x \le 2, 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

1) Find $P(1 < X < 1.5, 1 < Y < 1.5)$. Enter the answer correct to three decimal places

$$P(1 < x < 1.5, 1 < y < 1.5) = \int_1^{1.5} \int_1^{1.5} \frac{1}{3} \cdot dx \cdot dy$$

$$\Rightarrow \frac{1}{3} \int_1^{1.5} [x]_1^{1.5} \cdot dy$$

$$\Rightarrow \frac{1}{3} \int_1^{1.5} \frac{1}{2} \cdot dy$$

$$\Rightarrow \left(\frac{1}{3} \times \frac{1}{2}\right) [y]_1^{1.5}$$

$$\Rightarrow \frac{1}{12}$$

2) Find $P(1 < X < 1.5)$.

$$P(1 < x < 1.5) = \int_0^3 \int_1^{1.5} \frac{1}{3} \cdot dx \cdot dy$$

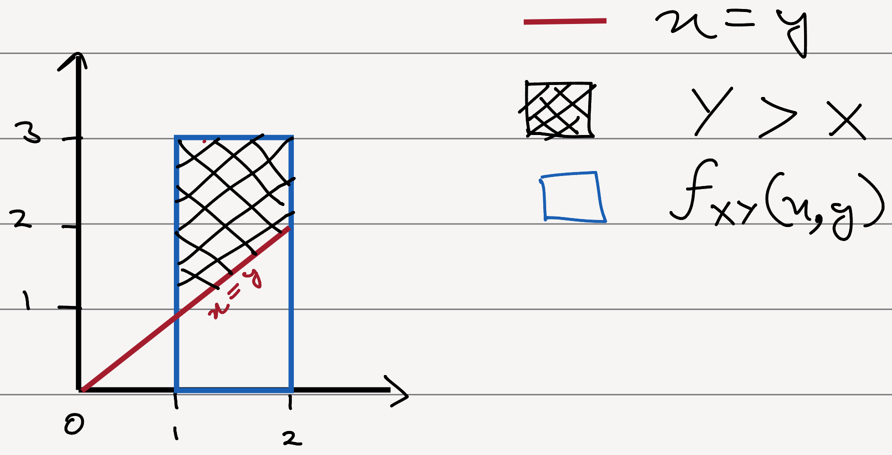
$$\Rightarrow \frac{1}{3} \int_0^3 [x]_1^{1.5} \cdot dy$$

$$\Rightarrow \frac{1}{3} \int_0^3 \frac{1}{2} \cdot dy$$

$$\Rightarrow \frac{1}{3} \times \frac{1}{2} [y]_0^3$$

$$\Rightarrow \frac{1}{2}$$

3) Find $P(Y > X)$.



Area of black region $= \int_1^2$ Upper curve − Lower Curve

$\Rightarrow \int_1^2 (3 - x)\,.dx$

$\Rightarrow \left[3x - \frac{x^2}{2}\right]_1^2$

$\Rightarrow \left[\left(6 - \frac{4}{2}\right) - \left(3 - \frac{1}{2}\right)\right]$

$\Rightarrow 4 - 3 + 0.5$

$\Rightarrow 1.5$

Area of blue region $= l \times b$

$\Rightarrow (3-0) \times (2-1)$

$\Rightarrow 3$

$\therefore P(Y > X) = \dfrac{\text{Area of black region}}{\text{Area of blue region}} = \dfrac{1.5}{3} = \underline{\underline{1/2}}$

Let $(X, Y) \sim \text{Uniform}(D)$, where $D := \{3X > Y, 0 < x < 1, y > 0\}$

4) Find the joint density of X and Y.

From set D we can see that $3X > Y$ and $y > 0$

$\Rightarrow 0 < y < 3X$ also,

$0 < x < 1$

Total area in given range will be,

$\Rightarrow \int_0^1 \int_0^{3x} 1 \cdot dy \cdot dx$

$\Rightarrow \int_0^1 [y]_0^{3x} \,.dx$

$\Rightarrow \int_0^1 3x\,.dx$

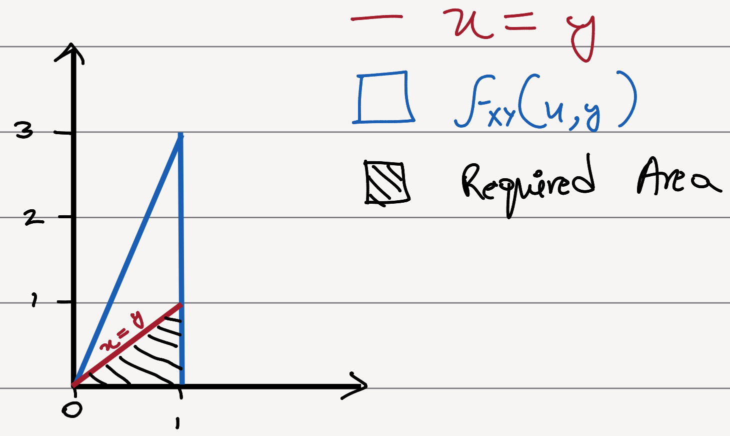
$\Rightarrow \left[\frac{3x^2}{2}\right]_0^1 = \frac{3}{2}$

$\Rightarrow \text{Density} = \dfrac{1}{|\text{Area}|} = \dfrac{1}{3/2} = \dfrac{2}{3}$

$$\therefore f_{XY}(x,y) = \begin{cases} 2/3 & \text{for } 0 < x < 1,\ 0 < y < 3x \\ 0 & \text{otherwise} \end{cases}$$

5) Find $P(X > Y)$.

$\Rightarrow P(Y<X) = \dfrac{\text{Black Area}}{\text{Total Area}}$

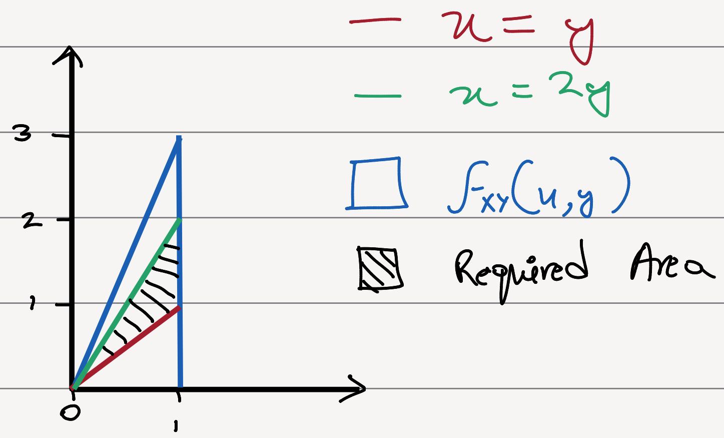


$\Rightarrow$ Black Area $= \frac{1}{2} \times b \times h$

$\Rightarrow \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

$\therefore P(Y<X) = \dfrac{1/2}{3/2} = \underline{\underline{\dfrac{1}{3}}}$

6) Find $P(Y < X < 2Y)$.



Area of Black Region,

$\int_0^1$ Upper Curve $-$ lower Curve

$\Rightarrow \int_0^1 (x - x/2) \cdot dx$

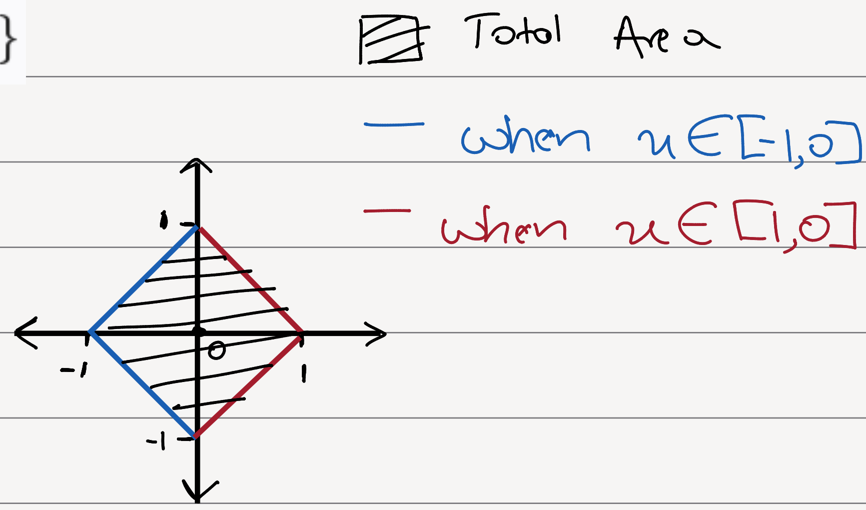
$\Rightarrow \int_0^1 x/2 \cdot dx$

$\Rightarrow \left[\dfrac{x^2}{4}\right]_0^1 = 1/4$

$\therefore P(Y < X < 2Y) = \dfrac{1/4}{3/2} = \underline{\underline{\dfrac{1}{6}}}$

Let $(X, Y) \sim \text{Uniform}(D)$, where $D := \{(x, y), \mid x \mid + \mid y \mid \leq 1\}$

7) Find the joint density of X and Y.



When $x \in [-1, 0]$,
$y \in [-1, 1]$

when $x \in [1, 0]$,
$y \in [-1, 1]$

Total Area $= 4\left(\frac{1}{2} \times b \times h\right)$

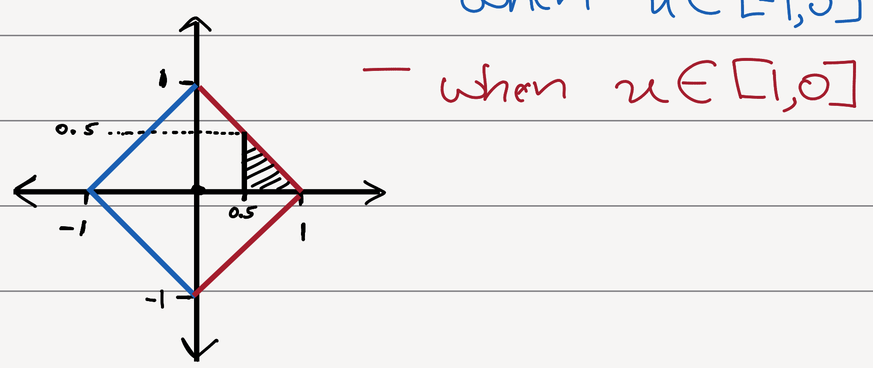
$\Rightarrow 4\left(\frac{1}{2} \times 1 \times 1\right)$

$\Rightarrow 2$

$\Rightarrow$ Density $= \dfrac{1}{|\text{Area}|} = \dfrac{1}{2}$

$\therefore f_{XY}(x,y) = \begin{cases} 1/2 & \text{when } x,y \in D \\ 0 & \text{otherwise} \end{cases}$

8) Find $P\left(\frac{1}{2} < X < 1, Y > 0\right)$. Enter the answer correct to 3 decimal points.



Required Area $= \frac{1}{2} \times b \times h$

$\Rightarrow \frac{1}{2} \times 0.5 \times 0.5$

$\therefore P(1/2 < X < 1, Y > 0) = \dfrac{\text{Required Area}}{\text{Total Area}}$

$\Rightarrow \dfrac{1/8}{2} = \underline{\underline{\dfrac{1}{16}}}$

9) Suppose X and Y have the joint PDF

$$f_{XY}(x,y) = \begin{cases} cx(x-y) & \text{for } 0 < x < 1, -x < y < x \\ 0 & \text{otherwise} \end{cases}$$

Find the value of c.

if range of x is $[0,1]$
and range of y is $[-x,x]$

$\Rightarrow c \int_0^1 \int_{-x}^{x} x^2 - xy = 1$

$\Rightarrow c \int_0^1 x^2 [y]_{-x}^{x} - x[y^2/2]_{-x}^{x} = 1$

$\Rightarrow c \int_0^1 x^2 [x - -x] - x\left[\frac{x^2}{2} - \left(\frac{(-x)^2}{2}\right)\right] = 1$

$\Rightarrow c \int_0^1 2x^3 - x[0] = 1$

$\Rightarrow 2c \left[\frac{x^4}{4}\right]_0^1 = 1$

$\Rightarrow \quad \frac{c}{2} = 1$

$\Rightarrow \quad \underline{\underline{c = 2}}$

Let X and Y be continuous random variables with joint density

$$f_{XY}(x,y) = \begin{cases} 6x^2y & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

10) Find $P\left(0 < X < \frac{1}{2}, 0 < Y < \frac{1}{2}\right)$. Enter the answer correct to 3 decimal points.

$\Rightarrow P(0 < X < 1/2, 0 < Y < 1/2) = \int_0^{1/2}\int_0^{1/2} 6x^2y \cdot dx \cdot dy$

$\Rightarrow 6\int_0^{1/2} y\left[\frac{x^3}{3}\right]_0^{1/2}$

$\Rightarrow 6\int_0^{1/2} y\left[\frac{1}{24}\right] \cdot dy$

$\Rightarrow \frac{1}{4}\left[\frac{y^2}{2}\right]_0^{1/2}$

$\Rightarrow \frac{1}{4} \times \frac{1}{8} = \underline{\underline{\frac{1}{32}}}$

11) Find $P(X \geq Y)$.

$P(X \geq Y) = 1 - P(X < Y)$

$P(X<Y) = \int_0^1\int_0^y 6x^2y \cdot dx \cdot dy$

$\Rightarrow 6\int_0^1 y\left[\frac{x^3}{3}\right]_0^y \cdot dy$

$\Rightarrow 2\int_0^1 y^4 \cdot dy$

$\Rightarrow 2\left[\frac{y^5}{5}\right]_0^1$

$\Rightarrow 2/5$

$\therefore P(X \geq Y) = 1 - 2/5 = \underline{\underline{3/5}}$