1) Let $X \sim \text{Uniform}\{1,2,3\}$. Let $(Y \mid X=1) \sim \text{Exp}(1)$, $(Y \mid X=2) \sim \text{Exp}(2)$ and $(Y \mid X=3) \sim \text{Exp}(3)$. Find the distribution of Y.

$$f_Y(y) = \sum_{x \in T_X} P_X(x)\, f_{Y|X=x}(y)$$

$$\Rightarrow \left(\frac{1}{3}\right)\text{Exp}(1) + \left(\frac{1}{3}\right)\text{Exp}(2) + \left(\frac{1}{3}\right)\text{Exp}(3)$$

$$\Rightarrow \frac{1}{3}\left[e^{-y} + 2e^{-2y} + 3e^{-3y}\right]$$

2) Let $X \sim \text{Binomial}\left(2, \frac{1}{2}\right)$. Let $(Y \mid X=0) \sim \text{Normal}(0,1)$, $(Y \mid X=1) \sim \text{Normal}(1,4)$ and $(Y \mid X=2) \sim \text{Normal}(2,9)$. Choose the correct option(s) from the following:

a) $f_{Y|X=2}(2) = \frac{1}{3\sqrt{2\pi}}$ ✓

b) $f_{Y|X=0}(2) = \frac{1}{\sqrt{2\pi}}e^{2}$

c) $f_Y(2) = \frac{1}{\sqrt{2\pi}}\left[e^{-2} + e^{-\frac{1}{8}} + \frac{1}{3}\right]$

d) $f_Y(2) = \frac{1}{4\sqrt{2\pi}}\left[e^{-2} + e^{-\frac{1}{8}} + \frac{1}{3}\right]$

Formula for,

Binomial Random Variable

$$\left({}^{n}C_{k}\right)\left(p^{k}\right)\left(1-p\right)^{n-k}$$

Normal distribution

$$\left(\frac{1}{\sigma\sqrt{2\pi}}\right)\text{Exp}\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^{2}\right)$$

a) $f_{Y|X=2}(2) \sim \text{Normal}(2,9)$

$$\Rightarrow \left(\frac{1}{\sigma\sqrt{2\pi}}\right)\text{Exp}\left(-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^{2}\right)$$

Here,

$\sigma = \sqrt{9} = 3$

$\mu = 2$

$y = 2$

$$\Rightarrow \left(\frac{1}{3\sqrt{2\pi}}\right)\text{Exp}\left(-\frac{1}{2}\left(\frac{2-2}{3}\right)^{2}\right)$$

$$\Rightarrow \frac{1}{3\sqrt{2\pi}}$$

b) $f_{Y|X=0}(2) \sim \text{Normal}(0,1)$

Here $\mu = 0$

$\sigma = \sqrt{1} = 1$

$y = 2$

$$\Rightarrow \left(\frac{1}{(1)\sqrt{2\pi}}\right)\text{Exp}\left(-\frac{1}{2}\left(\frac{2-0}{1}\right)^{2}\right)$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}e^{-2}$$

c) and d)

We know,

$$f_Y(y) = \sum_{x \in T_X} p_X(x)\, f_{Y|X=x}(y)$$

$$\Rightarrow \left({}^2C_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^2\right) \text{Normal}(0,1) + \left({}^2C_1 \left(\tfrac{1}{2}\right)^1 \left(\tfrac{1}{2}\right)^1\right) \text{Normal}(1,4)$$

$$+ \left({}^2C_2 \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^0\right) \text{Normal}(2,9)$$

$$\Rightarrow \left(\tfrac{1}{4}\right) \text{Normal}(0,1) + \left(\tfrac{1}{2}\right) \text{Normal}(1,4) + \left(\tfrac{1}{4}\right) \text{Normal}(2,9)$$

$$f_Y(2) = \left(\tfrac{1}{4}\right)\left(\frac{1}{\sqrt{2\pi}}\right) e^{-2} + \left(\tfrac{1}{2}\right)\left(\frac{1}{2\sqrt{2\pi}}\right) e^{-1/8} + \left(\tfrac{1}{4}\right)\left(\frac{1}{3\sqrt{2\pi}}\right) e^{0}$$

$$\Rightarrow \frac{1}{4\sqrt{2\pi}} \left[e^{-2} + e^{-1/8} + \frac{1}{3} \right]$$

3) Let $X \sim \text{Bernoulli}(0.5)$. Let $(Y \mid X = 0) \sim \text{Exp}(2)$ and $(Y \mid X = 1) \sim \text{Normal}(0, 1)$. Choose the correct option(s) from the following:

a) $P(X = 0 \mid Y = -1) = \dfrac{1}{\sqrt{2\pi}} e^{\frac{5}{2}}$

b) $P(X = 0 \mid Y = -1) = \sqrt{\dfrac{\pi}{2}} e^{\frac{5}{2}}$

c) $P(X = 1 \mid Y = -1) = 1$

d) $P(X = 1 \mid Y = 1) = \dfrac{e^{-\frac{1}{2}}}{e^{-\frac{1}{2}} + \sqrt{8\pi} e^{-2}}$

We know,

$$P(X = x \mid Y = y_0) = \frac{p_X(x)\, f_{Y|X=x}(y_0)}{f_Y(y_0)}$$

$$f_Y(y) = \left(\tfrac{1}{2}\right) \text{Exp}(2) + \left(\tfrac{1}{2}\right) \text{Normal}(0,1)$$

$$\Rightarrow \underbrace{e^{-2y}}_{\text{for all } Y > 0} + \underbrace{\frac{1}{2\sqrt{2\pi}}}_{\text{for all } Y} e^{-y^2/2}$$

a) and b)

$$P(X = 0 \mid Y = -1) = \frac{\left(\tfrac{1}{2}\right) 0}{0 + \left(\tfrac{1}{2\sqrt{2\pi}}\right) e^{-1/2}} = 0$$

[Exponential distribution is defined only when $y > 0$. Here $y < 0$.]

c)

$$P(X=1 \mid Y=-1) = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2\pi}}\right)e^{-1/2}}{0 + \frac{1}{2\sqrt{2\pi}}e^{-1/2}} = 1$$

d)

$$P(X=1 \mid Y=1) = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2\pi}}\right)e^{-1/2}}{e^{-2} + \left(\frac{1}{2\sqrt{2\pi}}\right)e^{-1/2}}$$

$$\Rightarrow \frac{\cancel{\left(\frac{1}{2\sqrt{2\pi}}\right)} e^{-1/2}}{\frac{2\sqrt{2\pi}\, e^{-2} + e^{-1/2}}{\cancel{2\sqrt{2\pi}}}}$$

$$\Rightarrow \frac{e^{-1/2}}{\sqrt{8\pi}e^{-2} + e^{-1/2}}$$

Let $Y = XZ, X \sim \text{Uniform}\{-1,0,1\}$ and $Z \sim \text{Normal}(1,2)$, where X and Z are independent.

4) Choose the correct option(s) from the following:

- ☐ $(Y \mid X=1) \sim \text{Normal}(1,2)$
- ☐ $(Y \mid X=-1) \sim \text{Normal}(-1,-2)$
- ☐ $(Y \mid X=-1) \sim \text{Normal}(-1,2)$
- ☐ $(Y \mid X=1) \sim \text{Normal}(1,-2)$