1) Let a random variable X be uniformly distributed over $[a, b]$ with expected value 6 and variance 4. Find the value of ab.

We know,

$$Var(x) = \frac{(b-a)^2}{12}, \quad E(x) = \frac{a+b}{2}$$

Given,

$E(x) = 6$, $Var(x) = 4$

$\Rightarrow (E(x))^2 = 36$

$\Rightarrow \left(\frac{a+b}{2}\right)^2 = 36$

$\Rightarrow a^2 + b^2 + 2ab = 144$

$\Rightarrow a^2 + b^2 = 144 - 2ab$ —— (1)

$Var(x) = 4$

$\Rightarrow \frac{(b-a)^2}{12} = 4$

$\Rightarrow a^2 + b^2 - 2ab = 48$

$a^2 + b^2 = 48 + 2ab$ —— (2)

$\Rightarrow 144 - 2ab = 48 + 2ab$ [From (1) and (2)]

$\Rightarrow 144 - 48 = 4ab$

$\Rightarrow 96 = 4ab$

$\Rightarrow \boxed{24 = ab}$

The probability density function of the time X (in minutes) between calls at the customer care center is given by

$$f(x) = \begin{cases} \frac{1}{4}.e^{\frac{-x}{4}} & \text{if } 0 \le x < \infty \\ 0 & \text{otherwise} \end{cases}$$

2) Find the expected time (in minutes) between calls.

$f(x) = \frac{1}{4} \cdot e^{-x/4}, \quad 0 \le x < \infty$

$\Rightarrow E[X] = \int_0^\infty x \cdot f(x) \cdot dx$

$\Rightarrow -1 \int -\frac{x}{4} \cdot e^{-x/4} \cdot dx$

Let $u = \frac{-x}{4}$ —— (1)

$\Rightarrow \frac{du}{dx} = \frac{-1}{4}$

$\Rightarrow \frac{dx}{du} = -4$

$\Rightarrow dx = -4.du$ —— (2)

$\Rightarrow -1 \int u.e^u . (-4.du)$ [From (1) and (2)]

$\Rightarrow \left[u \int e^u - \int (1)\, e^u\right] 4$

$\Rightarrow \left[u.e^u - e^u\right] -4$

$\Rightarrow 4\left[e^{-x/4} \cdot \frac{-x}{4} - e^{-x/4}\right]$

Applying limits 0 to ∞

$$\Rightarrow 4\left[\left(\frac{1}{e^{\infty}}\cdot\infty - \frac{1}{e^{\infty}}\right) - \left(e^{0}\cdot 0 - e^{0}\right)\right]$$

$$\Rightarrow 4\left[0 - (0 - 1)\right]$$

$$\Rightarrow \underline{\underline{4}}$$

3) Find the probability that time between calls exceeds the expected time.

$$P(x > 4) = 1 - P(x \leqslant 4)$$

For $P(x \leqslant 4)$

$$\Rightarrow \int_{0}^{4} \frac{1}{4}\cdot e^{-u/4}\cdot du$$

$$\Rightarrow \frac{1}{4}\int_{0}^{4} e^{-u/4}\cdot du$$

$$\Rightarrow \frac{1}{4}\left[\frac{e^{-u/4}}{-1/4}\right]_{0}^{4}$$

$$\Rightarrow -\left[e^{-u/4}\right]_{0}^{4}$$

$$\Rightarrow -\left[1/e - e^{0}\right]$$

$$\Rightarrow 1 - \frac{1}{e}$$

$$\therefore P(x > 4) = 1 - \left(1 - \frac{1}{e}\right)$$

$$\Rightarrow \underline{\underline{\frac{1}{e}}}$$

4) Let $X \sim \text{Exp}(5)$. Find $\text{Var}(X)$.

Variance of Exponential distribution $= \dfrac{1}{\lambda^2}$

Here $\lambda = 5$,

$$\Rightarrow \text{Var}(x) = \frac{1}{25} = 0.04$$

5) Let X be a continuous random variable. Probability that X takes a value within 2 standard deviations of the expected value is:

$$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$$

$$\Rightarrow P\left(\frac{\mu - 2\sigma - \mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{\mu + 2\sigma - \mu}{\sigma}\right)$$

$$\Rightarrow P\left(-2 \leq Z \leq 2\right)$$

$$\Rightarrow F_Z(2) - F_Z(-2) \simeq 0.95$$

6) The probability density function of a random variable X is given as

$$f_X(x) = \begin{cases} \frac{x^2}{3} & -1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the cumulative distribution function of X.

Differentiate each option and see which matches with PDF.

$$\Rightarrow \frac{d}{dx} \frac{x^3+1}{9}$$

$$\Rightarrow \frac{3x^2}{9}$$

$$\Rightarrow \frac{x^2}{3}$$

7) The probability density function of a random variable X is given as

$$f_X(x) = \begin{cases} \frac{20000}{x^3} & x > 100 \\ 0 & \text{otherwise} \end{cases}$$

Find the expected value of X. (Hint: $\int \frac{1}{x^2} dx = \frac{-1}{x} + c$ where c is a integration constant)

$$E[X] = \int_{100}^{\infty} x . f(x)$$

$$\Rightarrow 20000 \int_{100}^{\infty} x . \frac{1}{x^3} . dx$$

$$\Rightarrow 20000 \left[\frac{-1}{x} \right]_{100}^{\infty}$$

$$\Rightarrow 20000 \left[\frac{-1}{\infty} - \left(\frac{-1}{100}\right) \right]$$

$\Rightarrow \quad \underline{\underline{200}}$

8) The probability density function of a random variable X is given as

$$f_X(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the variance of X. Write your answer correct to two decimal points

$$E[X] = \int_0^1 x \cdot f(x)$$

$$\Rightarrow 2\int_0^1 x - x^2$$

$$\Rightarrow 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$\Rightarrow 2\left[\frac{1}{2} - \frac{1}{3}\right]_0^1$$

$$\Rightarrow \frac{2}{6} = \frac{1}{3}$$

$$E[X^2] = \int_0^1 x \cdot f(x)$$

$$\Rightarrow 2\int_0^1 x^2 - x^3$$

$$\Rightarrow 2\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$\Rightarrow \frac{2}{12} = \frac{1}{6}$$

$$\therefore Var(X) = E[X^2] - (E[X])^2$$

$$\Rightarrow \frac{1}{6} - \left(\frac{1}{3}\right)^2$$

$$\Rightarrow \frac{1}{6} - \frac{1}{9}$$

$$\Rightarrow \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$\Rightarrow \frac{1}{3}\left(\frac{1}{6}\right) = \underline{\underline{\frac{1}{18}}}$$

9) The probability density function of a random variable X is given as

$$f_X(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the expected value of X^2. Write your answer correct to two decimal places.

When $0 < x < 1$

$$\Rightarrow E[X^2] = \int_0^1 x^2 \cdot x$$

$$\Rightarrow \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

When $1 \leq x \leq 2$

$$\Rightarrow E[X^2] = \int_1^2 2x^2 - x^3$$

$$\Rightarrow \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2$$

$$\Rightarrow \left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right)$$

$$\Rightarrow \frac{16}{12} - \frac{5}{12}$$

$$\Rightarrow \frac{11}{12}$$

$$\therefore E[X^2] = \frac{11}{12} + \frac{1}{3}$$