1) Let $X$ be a continuous random variable with the following probability density function:

$$f_X(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{Otherwise} \end{cases}$$

Find the probability distribution function of $Y = X^2$.

$g(x) = y = x^2$

$\Rightarrow g'(x) = 2x$

To find inverse,

$\Rightarrow g(x) = x^2$

$\Rightarrow y = x^2$

$\Rightarrow x = y^2$

$\Rightarrow \sqrt{x} = y$

$\Rightarrow g^{-1}(x) = \sqrt{x}$

Also,

$g^{-1}(y) = \sqrt{y}$ — (1)

We know,

$$f_Y(y) = \frac{1}{g'(g^{-1}(y))} f_X(g^{-1}(y))$$

$$\Rightarrow \frac{1}{g'(\sqrt{y})} f_X(\sqrt{y}) \qquad [\quad]$$

$$\Rightarrow \left(\frac{1}{2\sqrt{y}}\right)\left(3(\sqrt{y})^2\right)$$

$$\Rightarrow \frac{3}{2}\sqrt{y}$$

$$\therefore f_Y(y) = \begin{cases} \frac{3}{2}\sqrt{y} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

2) Let $X \sim \text{Uniform}([-3, 3])$. Find the PDF of $|X|$.

All the negative values in range $[-3, 0]$ will become positive because of $|X|$.

$\Rightarrow$ PDF of $|X|$,

$X \sim \text{Uniform}[0, 3]$

3) Let $X \sim \text{Uniform}[-3, 2]$. Find the CDF of $|X|$.

$X \sim \text{Uniform}[-3, 2]$

We know,

$$F_X(x) = \frac{x - a}{b - a} \qquad [\text{When } X \sim \text{Uniform}[a, b]]$$

$$\Rightarrow \quad \frac{x-(-3)}{2-(-3)} \qquad [X \sim \text{Uniform}[-3,2]]$$

$$\Rightarrow \quad \frac{x+3}{5}$$

$$\therefore F_X(x) = \begin{cases} 0 & x < -3 \\ \frac{x+3}{5} & -3 \leq x \leq 2 \\ 1 & x > 2 \end{cases} \quad \text{— (1)}$$

Let $Y = |X|$,

When $X < 0$,

$\Rightarrow X \in [-3, 0]$

$\Rightarrow Y \in [0, 3]$ as $Y = |X|$

When $X > 0$,

$\Rightarrow X \in [0, 2]$

$\Rightarrow Y \in [0, 2]$ as $Y = |X|$

$\therefore$ Range of $Y = [0, 3]$

Here we will use the range given in the options.

$\Rightarrow Y \in [0, 2] \cup [2, 3]$

$F_Y(y) = P(Y \leq y)$

$\Rightarrow P(|X| \leq y)$

$\Rightarrow P(-y \leq X \leq y) = F_X(y) - F_X(-y)$

When $Y \in [0, 2]$

$X \in [-2, 2]$

In range $X \in [-2, 2]$,

CDF of $X = \frac{x+3}{5}$ [From (1)]

$\Rightarrow F_Y(y) = F_X(y) - F_X(-y)$

$$\Rightarrow \left(\frac{y+3}{5}\right) - \left(\frac{-y+3}{5}\right)$$

$\Rightarrow \quad \frac{2y}{5}$

When $Y \in [2,3]$

$X \in [-3,-2] \cup \{2\}$

$\Rightarrow F_Y(y) = F_X(y) - F_X(-y)$

$\Rightarrow F_X(2) - F_X(-y)$ [From ①]

$\Rightarrow \frac{2+3}{5} - \left(\frac{-y+3}{5}\right)$

$\Rightarrow \frac{2+y}{5}$

$$\therefore F_Y(y) = \begin{cases} 0 & y \leq 0 \\ 2y/5 & 0 < y \leq 2 \\ 2+y/5 & 2 < y \leq 3 \\ 1 & y > 3 \end{cases}$$

4) Let $X \sim \text{Uniform}[-3,2]$. Find the PDF of $|X|$.

When $Y \in [0,2]$

$f_Y(y) = \frac{d}{dy}\left(\frac{2y}{5}\right) = \frac{2}{5}$

When $Y \in [2,3]$

$f_Y(y) = \frac{d}{dy}\left(\frac{2+y}{5}\right) = \frac{1}{5}$

$$\therefore f_Y(y) = \begin{cases} 2/5 & 0 < y \leq 2 \\ 1/5 & 2 < y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

5) Let $X \sim \text{Exp}(\lambda)$. Find the PDF of $Y = X^3$.

$g(x) = Y = X^3$

$g'(x) = 3x^2$

For inverse,

$Y = X^3$

$\Rightarrow X = Y^3$

$\Rightarrow Y = \sqrt[3]{X}$

$\Rightarrow g^{-1}(x) = x^{1/3}$

Also,

$g^{-1}(y) = y^{1/3}$

We know,

$$f_Y(y) = \frac{1}{g'(g^{-1}(y))} f_X(g^{-1}(y))$$

$$\Rightarrow \frac{1}{3(y^{2/3})} \lambda e^{-\lambda y^{1/3}}$$

$$\therefore f_Y(y) = \begin{cases} \frac{\lambda}{(3)(y^{2/3})} \cdot e^{(-\lambda)(y^{1/3})} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

6) Let $X \sim \text{Normal}(\mu, \sigma^2)$. What will be the distribution of $aX + b$ where $a$ and $b$ are constants?

Let $Y = aX + b$

$$\Rightarrow E[Y] = E[aX + b]$$
$$\Rightarrow E[aX] + b$$
$$\Rightarrow (a)(E[X]) + b$$
$$\Rightarrow (a)(\mu) + b$$

$$Var(Y) = Var[aX + b]$$
$$\Rightarrow Var[aX]$$
$$\Rightarrow (a^2)(Var[X])$$
$$\Rightarrow (a^2)(\sigma^2)$$

$\therefore$ New distribution will be,

$$\text{Normal}(a\mu + b, a^2\sigma^2)$$

7) Let $X$ be a continuous random variable with probability density function

$$f_X(x) = \begin{cases} \frac{x}{12} & 1 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of $Y = 2X - 3$.

$$g(x) = Y = 2X - 3$$

$$g'(x) = 2$$

For inverse,

$$Y = 2X - 3$$
$$\Rightarrow X = 2Y - 3$$
$$\Rightarrow \frac{X+3}{2} = Y$$
$$\Rightarrow g^{-1}(x) = \frac{x+3}{2}$$

Also,

$$g^{-1}(y) = \frac{y+3}{2}$$

We know,

$$f_Y(y) = \frac{1}{g'(g^{-1}(y))} f_X(g^{-1}(y))$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{1}{12}\left(\frac{y+3}{2}\right)\right)$$

$$\Rightarrow \quad \frac{y+3}{48}$$

Range of $Y = [2(1)-3,\ 2(5)-3]$ $\quad$ [Note: $x \in [1, 5]$]

$\Rightarrow [-1, 7]$

$$\therefore\ f_Y(y) = \begin{cases} \dfrac{y+3}{48} & -1 < y < 7 \\ 0 & \text{otherwise} \end{cases}$$